

EXCELLENT NORMAL LOCAL DOMAINS AND EXTENSIONS OF KRULL DOMAINS

WILLIAM HEINZER, CHRISTEL ROTTHAUS, AND SYLVIA WIEGAND

This paper is dedicated to Hans-Bjørn Foxby.

ABSTRACT. We consider properties of extensions of Krull domains such as flatness that involve behavior of extensions and contractions of prime ideals. Let (R, \mathbf{m}) be an excellent normal local domain with field of fractions K , let y be a nonzero element of \mathbf{m} and let R^* denote the (y) -adic completion of R . For elements τ_1, \dots, τ_s of yR^* that are algebraically independent over R , we construct two associated Krull domains: an intersection domain $A := K(\tau_1, \dots, \tau_s) \cap R^*$ and its approximation domain B ; see Setting 2.2.

If in addition R is countable with $\dim R \geq 2$, we prove that there exist elements $\tau_1, \dots, \tau_s, \dots$ as above such that, for each $s \in \mathbb{N}$, the extension $R[\tau_1, \dots, \tau_s] \hookrightarrow R^*[1/y]$ is flat; equivalently, $B = A$ and A is Noetherian. Using this result we establish the existence of a normal Noetherian local domain B such that: B dominates R ; B has (y) -adic completion R^* ; and B contains a height-one prime ideal \mathbf{p} such that $R^*/\mathbf{p}R^*$ is not reduced. Thus B is not a Nagata domain and hence is not excellent.

We present several theorems involving the construction. These theorems yield examples where $B \subsetneq A$ and A is Noetherian while B is not Noetherian; and other examples where $B = A$ and A is not Noetherian.

1. INTRODUCTION

About twenty years ago Judy Sally gave an expository talk on the following question:

Question 1.1. What rings lie between a Noetherian integral domain S and its field of fractions $\mathcal{Q}(S)$?

We are inspired by work of Shreeram Abhyankar such as that in his paper [1] to ask the following related question: ¹

Question 1.2. Let I be an ideal of a Noetherian integral domain R and let R^* denote the I -adic completion of R . What rings lie between R and R^* ?

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¹Ram's work demonstrates the vastness of power series rings. The authors have fond memories of many pleasant conversations with him concerning power series.

A wide variety of integral domains fit the descriptions of both Questions 1.1 and 1.2. Let (R, \mathbf{m}) be an excellent normal local domain and let S be a polynomial ring in finitely many variables over R . In work over a number of years related to these questions, the authors have been developing techniques for constructing examples that are birational extensions of S and also subrings of an ideal-adic completion of R . Classical constructions of Noetherian integral domains with interesting properties, such as failure to be a Nagata ring, have been given by Akizuki, Schmidt, Nagata and others, [2], [10], [7]. We recall that a ring A is a *Nagata ring* if A is Noetherian and if the integral closure of A/P in L is finite over A/P , for every prime ideal P of A and every field L finite algebraic over the field of fractions of A/P , [6, page 264].

We often use in our construction the completion of an excellent normal local domain (R, \mathbf{m}) with respect to a principal ideal yR , where y is a nonzero nonunit of R . The (y) -adic completion R^* of R may be regarded as either an inverse limit or as a homomorphic image of a formal power series ring $R[[z]]$ over R . Thus we have

$$R^* = \varprojlim_n \left(\frac{R}{y^n R} \right) = \frac{R[[z]]}{(z - y)R[[z]]}.$$

An element τ of R^* has an expression as a power series in y with coefficients in R . It is often the case that there exist elements τ_1, \dots, τ_n in R^* that are algebraically independent over R . An elementary cardinality argument shows this is always the case if R is countable. Assume that τ_1, \dots, τ_n in R^* are algebraically independent over R . By modifying τ_i by an element in R , we may assume that each $\tau_i \in yR^*$. Let $S := R[\tau_1, \dots, \tau_n]$. Then S is both a subring of R^* and a polynomial ring in n variables over R . Although the expression for the τ_i as power series in y with coefficients in R is not unique, we use it to construct an integral domain B that is a directed union of localized polynomial rings over R .

The construction we consider associates with R and τ_1, \dots, τ_n the following two integral domains:

- (1) an intersection domain $A := \mathcal{Q}(S) \cap R^*$, and
- (2) an integral domain $B \subseteq A$ that approximates A .

The integral domain B is a directed union of localized polynomial rings in n variables over R . The rings B and A are birational extensions of the polynomial ring S and subrings of R^* . Thus they fit the descriptions of both Questions 1.1 and 1.2. For integral domains A and B obtained as above, we ask:

Questions 1.3. For a given excellent normal local domain (R, \mathbf{m}) and elements τ_1, \dots, τ_n as above, what properties do the constructed rings A and B have, and what criteria determine these properties?

Our work in this article concerning Questions 1.3 focuses primarily on the case where the base ring R is an excellent normal local domain. The intersection domain $A = \mathcal{Q}(S) \cap R^*$ may fail to be Noetherian even though R , and therefore R^* , is an excellent normal local domain. However, the intersection domain A is always a Krull domain, and the (y) -adic completion of A is R^* . Thus, in order to present an iterative procedure, in Section 2 we present many of the properties we study with the following Krull domain setting:

Setting 1.4. Let (T, \mathbf{n}) be a local Krull domain with field of fractions $\mathcal{Q}(T)$. Assume that $y \in \mathbf{n}$ is a nonzero element such that the (y) -adic completion (T^*, \mathbf{n}^*) of T is an analytically normal Noetherian local domain. Since the \mathbf{n} -adic completion of T is the same as the \mathbf{n}^* -adic completion of T^* , it follows that the \mathbf{n} -adic completion \widehat{T} of T is also a normal Noetherian local domain. Let $\mathcal{Q}(T^*)$ denote the field of fractions of T^* . Since T^* is Noetherian, \widehat{T} is faithfully flat over T^* and we have $T^* = \widehat{T} \cap \mathcal{Q}(T^*)$. Therefore $\mathcal{Q}(T) \cap T^* = \mathcal{Q}(T) \cap \widehat{T}$. Assume that $T = \mathcal{Q}(T) \cap T^*$, and let d denote the dimension of the Noetherian domain T^* . It follows that d is also the dimension of \widehat{T} .

Let $\tau_1 \dots, \tau_s$ be elements of yT^* that are algebraically independent over T . We consider the extensions

$$T \hookrightarrow T[\tau_1, \dots, \tau_s] \hookrightarrow A := \mathcal{Q}(T)(\tau_1, \dots, \tau_s) \cap T^* \hookrightarrow T^*.$$

In particular, the following map is critical:

$$(1.4.0) \quad \varphi: T[\tau_1, \dots, \tau_s] \hookrightarrow T^*[1/y].$$

The intersection ring $A = \mathcal{Q}(T)(\tau_1, \dots, \tau_s) \cap T^*$ and its approximating ring B may be Noetherian or not, or excellent or not. Examples 4.3 and 4.8 demonstrate that one may have $B = A$, or $B \subsetneq A$. Properties of the rings B and A are related to properties of the map φ of Equation 1.4.0; this is illustrated by two conclusions of Theorem 2.6:

- (1) B is Noetherian if and only if φ is flat.
- (2) If B is Noetherian, then $B = A$.

In general, by Theorem 2.9, we have $B = A$ if and only if φ satisfies the *weak flatness* property in Definitions 2.7 below. We describe in Section 2 the construction of B .

The rings $T[\tau_1, \dots, \tau_s]$ and $T^*[1/y]$ in Equation 1.4.0 are Krull domains as are also the constructed rings $B \hookrightarrow A$. We demonstrate connections between properties of the extension of Krull domains defined by the map φ in Equation 1.4.0, and properties of the Krull domains B and A . Some of our results hold for Krull domains as in Setting 1.4; for others we restrict to the case where $T = R$ is an excellent normal local domain.

It is useful to give a name for the elements τ_1, \dots, τ_s in case the map φ of Equation 1.4.0 is flat.

Definition 1.5. Assume Setting 1.4; thus T is a local Krull domain and y is a nonzero nonunit element of T such that the (y) -adic completion T^* of T is an analytically normal Noetherian local domain with $T = \mathcal{Q}(T) \cap T^*$. Elements $\tau_1, \dots, \tau_s \in yT^*$ that are algebraically independent over T are said to be *primarily limit-intersecting* in y over T provided the inclusion map

$$\varphi : T[\tau_1, \dots, \tau_s] \hookrightarrow T^*[1/y]$$

is flat. An infinite set $\{\tau_i\}_{i=1}^\infty$ of elements in yT^* that are algebraically independent over T is said to be *primarily limit-intersecting* in y over T if for each positive integer s , the elements τ_1, \dots, τ_s are primarily limit-intersecting in y over T .

It is natural to ask about the existence of primarily limit-intersecting elements:

Question 1.6. Let R be an excellent normal local domain with $\dim R = d \geq 2$, let y be a nonzero element in the maximal ideal \mathfrak{m} of R , and let R^* be the (y) -adic completion of R . Under what conditions on R do there exist elements that are primarily limit-intersecting in y over R ?

With notation as in Question 1.6, we describe in Theorem 3.5 and Remark 3.7 necessary and sufficient conditions that an element $\tau \in yR^*$ be primarily limit-intersecting in y over R . If R is countable, we prove in Theorem 3.12 the existence of an infinite sequence $\tau_1, \dots, \tau_s, \dots \in yR^*$ of elements that are primarily limit-intersecting in y over R . We show in Theorem 3.13 that in general for an element $\eta \in yR^*$ that is primarily limit-intersecting in y over R , the constructed Noetherian domain

$$B = A = R^* \cap \mathcal{Q}(R[\eta])$$

may fail to be excellent.

In Section 4 we present two theorems involving the construction. These theorems yield examples where $B \subsetneq A$ and A is Noetherian while B is not Noetherian; and other examples where $B = A$ and A is not Noetherian. We describe several examples obtained by iteration of the construction considered in Section 3.

2. BASIC PROPERTIES AND THE APPROXIMATION DOMAIN

In this section we give background and terminology. We generally assume Setting 1.4 in this section. First we illustrate the construction of an intersection domain A and develop the terminology necessary for an approximation domain B in what we consider the easiest example that can be constructed.

Example 2.1. The “easiest” example. Let k be a field, for example, $k = \mathbb{Q}$, the rational numbers, and let x be an indeterminate over k . Let $R = k[x]_{(x)}$ and $R^* = k[[x]]$, the power series ring in x over k and the x -adic completion of R . Let $\tau \in xk[[x]]$ be algebraically independent over R ; for example, if $k = \mathbb{Q}$, we could take $\tau = e^x - 1$. Define A , the *intersection domain* associated to τ over R by

$$A = k(x, \tau) \cap k[[x]].$$

In this case A is a rank-one discrete valuation ring (DVR) because it is the intersection of the DVR $k[[x]]$ with a subfield of $k((x))$ that is not contained in k . Thus A is a Noetherian one-dimensional regular local ring (RLR) and the unique maximal ideal is xA .

We apply approximation techniques to more precisely describe the elements that are in A . In order to define an approximation domain B that goes with A , write

$$\tau = \sum_{i=1}^{\infty} a_i x^i,$$

where the $a_i \in k$. Define $\tau_0 = \tau$. For each $n \in \mathbb{N}$, define the n^{th} *endpiece* of τ , denoted τ_n , and define rings U_n and B_n by

$$\tau_n = \sum_{i=n+1}^{\infty} a_i x^{i-n}, \quad U_n = k[x, \tau_n] \quad \text{and} \quad B_n = k[x, \tau_n]_{(x, \tau_n)}.$$

Set

$$U = \cup_{n=0}^{\infty} U_n \quad \text{and} \quad B = \cup_{n=0}^{\infty} B_n.$$

It is straightforward to show that $A = B$ in Example 2.1; see [4, Chapter 6]. Since the extension $k[x, \tau_n] \hookrightarrow A$ does not satisfy the dimension inequality [6, p. 119], the ring $A = B$ is *not* the localization of a finitely generated algebra over k .

In general the intersection of a normal Noetherian domain with a subfield of its field of fractions is a Krull domain, but need not be Noetherian. A directed union of normal Noetherian domains may be a non-Noetherian Krull domain. Thus, in order to be able to iterate our construction, we consider a local Krull domain (T, \mathbf{n}) that is not assumed to be Noetherian, but is assumed to have a Noetherian completion. To distinguish from the Noetherian hypothesis on R , we let T denote the base domain.

The construction of the approximation domain.

Setting and Notation 2.2. Let (T, \mathbf{n}) be a local Krull domain with field of fractions F . Assume there exists a nonzero element $y \in \mathbf{n}$ such that the y -adic completion $\widehat{(T, (y))} := (T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain. It then follows that the \mathbf{n} -adic completion \widehat{T} of T is also a normal Noetherian local domain, since the \mathbf{n} -adic completion of T is the same as the \mathbf{n}^* -adic

completion of T^* . Since T^* is Noetherian, if F^* denotes the field of fractions of T^* , then $T^* = \widehat{T} \cap F^*$. Therefore $F \cap T^* = F \cap \widehat{T}$. Let d denote the dimension of the Noetherian domain T^* . It follows that d is also the dimension of \widehat{T} .²

(1) Assume that $T = F \cap T^* = F \cap \widehat{T}$, or equivalently by Proposition 2.8.1, that T^* and \widehat{T} are weakly flat over T .

(2) Let $\widehat{T}[1/y]$ denote the localization of \widehat{T} at the powers of y , and similarly, let $T^*[1/y]$ denote the localization of T^* at the powers of y . The domains $\widehat{T}[1/y]$ and $T^*[1/y]$ have dimension $d - 1$.

(3) Let $\tau_1, \dots, \tau_s \in \mathbf{n}^*$ be algebraically independent over F .

(4) For each i with $1 \leq i \leq s$, we have an expansion $\tau_i := \sum_{j=1}^{\infty} c_{ij} y^j$ where $c_{ij} \in T$.

(5) For each $n \in \mathbb{N}$ and each i with $1 \leq i \leq s$, we define the n^{th} -endpiece τ_{in} of τ_i with respect to y as in Example 2.1:

$$(2.2.0) \quad \tau_{in} := \sum_{j=n+1}^{\infty} c_{ij} y^{j-n}.$$

Thus we have $\tau_{in} = y\tau_{i,n+1} + c_{i,n+1}y$.

(6) For each $n \in \mathbb{N}$, we define $B_n := T[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{n}, \tau_{1n}, \dots, \tau_{sn})}$. In view of (5), we have $B_n \subseteq B_{n+1}$ and B_{n+1} dominates B_n for each n . We define

$$B := \varinjlim_{n \in \mathbb{N}} B_n = \bigcup_{n=1}^{\infty} B_n, \quad \text{and} \quad A := F(\tau_1, \dots, \tau_s) \cap \widehat{T}.$$

Thus, B and A are local Krull domains and A birationally dominates B . We are especially interested in conditions which imply that $B = A$.

(7) Let A^* denote the y -adic completion of A and let B^* denote the y -adic completion of B .

Remarks 2.3. The definitions of B and B_n are independent of representations for τ_1, \dots, τ_s as power series in y with coefficients in T ; see [4, Proposition 21.6].

Properties of the construction.

The following theorem is proved in [4, Theorem 21.7].

Theorem 2.4. *Assume the setting and notation of (2.2). Then the intermediate rings B_n , B and A have the following properties:*

- (1) $yA = yT^* \cap A$ and $yB = yA \cap B = yT^* \cap B$. More generally, for every $t \in \mathbb{N}$, we have $y^t A = y^t T^* \cap A$ and $y^t B = y^t A \cap B = y^t T^* \cap B$.
- (2) $T/y^t T = B/y^t B = A/y^t A = T^*/y^t T^*$, for each positive integer t .

²If T is Noetherian, then d is also the dimension of T . However, if T is not Noetherian, then the dimension of T may be greater than d . This is illustrated by taking T to be the ring B of Example 4.3.

- (3) Every ideal of T, B or A that contains y is finitely generated by elements of T . In particular, the maximal ideal \mathfrak{n} of T is finitely generated, and the maximal ideals of B and A are $\mathfrak{n}B$ and $\mathfrak{n}A$.
- (4) For every $n \in \mathbb{N}$: $yB \cap B_n = (y, \tau_{1n}, \dots, \tau_{sn})B_n$, an ideal of B_n of height $s + 1$.
- (5) Let $P \in \text{Spec}(A)$ be minimal over yA , and let $Q = P \cap B$ and $W = P \cap T$. Then $T_W \subseteq B_Q = A_P$, and all three localizations are DVRs.
- (6) For every $n \in \mathbb{N}$, $B[1/y]$ is a localization of B_n , i.e., for each $n \in \mathbb{N}$, there exists a multiplicatively closed subset S_n of B_n such that $B[1/y] = S_n^{-1}B_n$.
- (7) $B = B[1/y] \cap B_{\mathfrak{q}_1} \cap \dots \cap B_{\mathfrak{q}_r}$, where $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ are the prime ideals of B minimal over yB .

The next theorem from [4, Theorem 21.8] is also useful in the sequel.

Theorem 2.5. *With the setting and notation of (2.2), the intermediate rings A and B have the following properties:*

- (1) A and B are local Krull domains.
- (2) $B \subseteq A$, with A dominating B .
- (3) $A^* = B^* = T^*$.
- (4) If B is Noetherian, then $B = A$.

Moreover, if T is a unique factorization domain (UFD) and y is a prime element of T , then B is a UFD.

We use the following theorem [4, Theorem 21.13] to establish the Noetherian property.

Theorem 2.6. *Assume the notation of Setting 2.2. Thus (T, \mathfrak{n}) is a local Krull domain with field of fractions F , and $y \in \mathfrak{n}$ is such that the (y) -adic completion (T^*, \mathfrak{n}^*) of T is an analytically normal Noetherian local domain and $T = T^* \cap F$. For elements $\tau_1, \dots, \tau_s \in \mathfrak{n}^*$ that are algebraically independent over T , the following are equivalent:*

- (1) The extension $T[\tau_1, \dots, \tau_s] \hookrightarrow T^*[1/y]$ is flat.
- (2) The elements τ_1, \dots, τ_s are primarily limit-intersecting in y over T .
- (3) The intermediate rings A and B are equal and are Noetherian.
- (4) The constructed ring B is Noetherian.

Moreover, if these equivalent conditions hold, then the Krull domain T is Noetherian.

We consider the following properties of an extension of Krull domains.

Definitions 2.7. Let $S \hookrightarrow T$ be an extension of Krull domains.

- (1) We say that the extension $S \hookrightarrow T$ is *weakly flat*, or that T is *weakly flat* over S , if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.
- (2) We say that the extension $S \hookrightarrow T$ is *height-one preserving*, or that T is a *height-one preserving* extension of S , if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.
- (3) For $d \in \mathbb{N}$, we say that $\varphi : S \hookrightarrow T$ *satisfies LF_d* (*locally flat in height d*), if, for each $P \in \text{Spec } T$ with $\text{ht } P \leq d$, the composite map $S \rightarrow T \rightarrow T_P$ is flat.

The condition LF_1 is equivalent to weak flatness. If $\dim T \leq d$, then the condition LF_d is equivalent to flatness. Proposition 2.8 demonstrates the relevance of the weak flatness property for an extension of Krull domains.

Proposition 2.8. [4, Corollary 12.4] *Let $\varphi : S \hookrightarrow T$ be an extension of Krull domains and let F denote the field of fractions of S .*

- (1) *Assume that $PT \neq T$ for every height-one prime ideal P of S . Then $S \hookrightarrow T$ is weakly flat $\iff S = F \cap T$.*
- (2) *If $S \hookrightarrow T$ is weakly flat, then φ is height-one preserving and, moreover, for every height-one prime ideal P of S with $PT \neq T$, there is a height-one prime ideal Q of T with $Q \cap S = P$.*

Theorem 2.9 states that weak flatness of the map φ of Equation 1.4.0 is equivalent to equality of the intersection domain A with its approximation domain B .

Theorem 2.9. [4, Theorem 21.14] *Assume the notation of Setting 2.2. Thus (T, \mathfrak{n}) is a local Krull domain with field of fractions F , and $y \in \mathfrak{n}$ is such that the (y) -adic completion (T^*, \mathfrak{n}^*) of T is an analytically normal Noetherian local domain and $T = T^* \cap F$. For elements $\tau_1, \dots, \tau_s \in \mathfrak{n}^*$ that are algebraically independent over T , the following are equivalent:*

- (1) *The intersection domain A is equal to its approximation domain B .*
- (2) *The map $\varphi : T[\tau_1, \dots, \tau_s] \longrightarrow T^*[1/y]$ is weakly flat.*
- (3) *The map $B \longrightarrow T^*[1/y]$ is weakly flat.*
- (4) *The map $B \longrightarrow T^*$ is weakly flat.*

3. PRIMARILY LIMIT-INTERSECTING ELEMENTS

In this section, we establish the existence of primarily limit-intersecting elements over countable excellent normal local domains.

We use Corollary 3.2 and Lemma 3.3 in the proof of Theorem 3.8.

Theorem 3.1. [4, Theorem 11.3] *Let (R, \mathbf{m}) , (S, \mathbf{n}) and (T, ℓ) be Noetherian local rings, and assume there exist local homomorphisms:*

$$R \longrightarrow S \longrightarrow T,$$

such that

- (i) *$R \rightarrow T$ is flat and $T/\mathbf{m}T$ is Cohen-Macaulay.*
- (ii) *$R \rightarrow S$ is flat and $S/\mathbf{m}S$ is a regular local ring.*

Then the following statements are equivalent:

- (1) *$S \rightarrow T$ is flat.*
- (2) *For each prime ideal \mathbf{w} of T , we have $\text{ht}(\mathbf{w}) \geq \text{ht}(\mathbf{w} \cap S)$.*
- (3) *For each prime ideal \mathbf{w} of T such that \mathbf{w} is minimal over $\mathbf{n}T$, we have $\text{ht}(\mathbf{w}) \geq \text{ht}(\mathbf{n})$.*

Since flatness is a local property, the following corollary is immediate.

Corollary 3.2. *Let R , S and T be Noetherian rings, and assume there exist ring homomorphisms $R \rightarrow S \rightarrow T$. If the map $R \rightarrow T$ is flat with Cohen-Macaulay fibers and the map $R \rightarrow S$ is flat with regular fibers, then the following two statements are equivalent:*

- (1) *The map $S \rightarrow T$ is flat,*
- (2) *For each prime ideal P of T , we have $\text{ht}(P) \geq \text{ht}(P \cap S)$.*

To establish the existence of primarily limit-intersecting elements, we use the following prime avoidance lemma; see the articles [3], [11], [13] and the book [5, Lemma 14.2] for other prime avoidance results involving countably infinitely many prime ideals.

Lemma 3.3. [4, Lemma 22.10] *Let (T, \mathbf{n}) be a Noetherian local domain that is complete in the (y) -adic topology, where y is a nonzero element of \mathbf{n} . Let \mathcal{U} be a countable set of prime ideals of T such that $y \notin P$ for each $P \in \mathcal{U}$, and fix an arbitrary element $t \in \mathbf{n} \setminus \mathbf{n}^2$. Then there exists an element $a \in y^2T$ such that $t - a \notin \bigcup \{P : P \in \mathcal{U}\}$.*

Proof. We may assume there are no inclusion relations among the $P \in \mathcal{U}$. We enumerate the prime ideals in \mathcal{U} as $\{P_i\}_{i=1}^\infty$. We choose $b_2 \in T$ so that $t - b_2y \notin P_1$ as follows: (i) if $t \in P_1$, let $b_2 = 1$. Since $y \notin P_1$, we have $t - y^2 \notin P_1$. (ii) if $t \notin P_1$, let b_2 be a nonzero element of P_1 . Then $t - b_2y^2 \notin P_1$. Assume by induction that we have found b_2, \dots, b_n in T such that

$$t - cy^2 := t - b_2y^2 - \dots - b_ny^n \notin P_1 \cup \dots \cup P_{n-1}.$$

We choose $b_{n+1} \in T$ so that $t - cy^2 - b_{n+1}y^{n+1} \notin \bigcup_{i=1}^n P_i$ as follows: (i) if $t - cy^2 \in P_n$, let $b_{n+1} \in (\prod_{i=1}^{n-1} P_i) \setminus P_n$. (ii) if $t - cy^2 \notin P_n$, let b_{n+1} be any nonzero

element in $\prod_{i=1}^n P_i$. Hence in either case there exists $b_{n+1} \in T$ so that

$$t - b_2 y^2 - \cdots - b_{n+1} y^{n+1} \notin P_1 \cup \cdots \cup P_n.$$

Since T is complete in the (y) -adic topology, the Cauchy sequence

$$\{b_2 y^2 + \cdots + b_n y^n\}_{n=2}^\infty$$

has a limit $a \in \mathfrak{n}^2$. Since T is Noetherian and local, every ideal of T is closed in the (y) -adic topology. Hence, for each integer $n \geq 2$, we have

$$t - a = (t - b_2 y^2 - \cdots - b_n y^n) - (b_{n+1} y^{n+1} + \cdots),$$

where $t - b_2 y^2 - \cdots - b_n y^n \notin P_{n-1}$ and $(b_{n+1} y^{n+1} + \cdots) \in P_{n-1}$. We conclude that $t - a \notin \bigcup_{i=1}^\infty P_i$. \square

The existence of one primarily limit-intersecting element.

We use the following setting to describe necessary and sufficient conditions for an element to be primarily limit-intersecting.

Setting 3.4. Let (R, \mathfrak{m}) be a d -dimensional excellent normal local domain with $d \geq 2$, let y be a nonzero element of \mathfrak{m} and let R^* denote the (y) -adic completion of R . Let t be a variable over R , let $S := R[t]_{(\mathfrak{m}, t)}$, and let S^* denote the I -adic completion of S , where $I := (y, t)S$. Then $S^* = R^*[[t]]$ is a $(d+1)$ -dimensional normal Noetherian local domain with maximal ideal $\mathfrak{n}^* := (\mathfrak{m}, t)S^*$. For each element $a \in y^2 S^*$, we have $S^* = R^*[[t]] = R^*[[t - a]]$ since R^* is complete in the (a) -adic topology. Let $\lambda_a : S^* \rightarrow R^*$ denote the canonical homomorphism $S^* \rightarrow S^*/(t - a)S^* = R^*$, and let $\tau_a = \lambda_a(t) = \lambda_a(a)$. Consider the set

$$\mathcal{U} := \{P^* \in \text{Spec } S^* \mid \text{ht}(P^* \cap S) = \text{ht } P^*, \text{ and } y \notin P^*\}.$$

Since $S \hookrightarrow S^*$ is flat and thus satisfies the going-down property, the set \mathcal{U} can also be described as the set of all $P^* \in \text{Spec } S^*$ such that $y \notin P^*$ and P^* is minimal over PS^* for some $P \in \text{Spec } S$; see [6, Theorem 15.1]

Theorem 3.5. *With the notation of Setting 3.4, the element τ_a is primarily limit-intersecting in y over R if and only if $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} S = R[t]_{(\mathfrak{m}, t)} & \xrightarrow{\subseteq} & S^* = R^*[[t]] & \xrightarrow{\subseteq} & S^*[1/y] \\ \lambda_0 \downarrow & & \lambda_a \downarrow & & \\ R & \xrightarrow{\subseteq} & R_1 = R[\tau_a]_{(\mathfrak{m}, \tau_a)} & \longrightarrow & R^* & \xrightarrow{\subseteq} & R^*[1/y]. \end{array}$$

Diagram 3.5.0

The map λ_0 denotes the restriction of λ_a to S .

Assume that τ_a is primarily limit-intersecting in y over R . Then τ_a is algebraically independent over R and λ_0 is an isomorphism. If $t - a \in P^*$ for some $P^* \in \mathcal{U}$, we prove that $\varphi : R_1 \rightarrow R^*[1/y]$ is not flat. Let $Q^* := \lambda_a(P^*)$. We have $\text{ht } Q^* = \text{ht } P^* - 1$, and $y \notin P^*$ implies $y \notin Q^*$. Let $P := P^* \cap S$ and $Q := Q^* \cap R_1$. Commutativity of Diagram 3.5.0 and λ_0 an isomorphism imply that $\text{ht } P = \text{ht } Q$. Since $P^* \in \mathcal{U}$, we have $\text{ht } P = \text{ht } P^*$. It follows that $\text{ht } Q > \text{ht } Q^*$. This implies that $\varphi : R_1 \rightarrow R^*[1/y]$ is not flat.

For the converse, assume that $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$. Since $a \in y^2 S^*$ and S^* is complete in the (y, t) -adic topology, we have $S^* = R^*[[t]] = R^*[[t - a]]$. Thus

$$\mathfrak{p} := \ker(\lambda_a) = (t - \tau_a)S^* = (t - a)S^*$$

is a height-one prime ideal of S^* . Since $y \in R$ and $\mathfrak{p} \cap R = (0)$, we have $y \notin \mathfrak{p}$.

Since $t - a$ is outside every element of \mathcal{U} , we have $\mathfrak{p} \notin \mathcal{U}$. Since \mathfrak{p} does not fit the condition of \mathcal{U} , we have $\text{ht}(\mathfrak{p} \cap S) \neq \text{ht } \mathfrak{p} = 1$, and so, by the faithful flatness of $S \hookrightarrow S^*$, $\mathfrak{p} \cap S = (0)$. Therefore the map $\lambda_0 : S \rightarrow R_1$ has trivial kernel, and so λ_0 is an isomorphism. Thus τ_a is algebraically independent over R .

Since R is excellent and R_1 is a localized polynomial ring over R , the hypotheses of Corollary 3.2 are satisfied for the extension $R \hookrightarrow R_1 \hookrightarrow R^*[1/y]$. It follows that the element τ_a is primarily limit-intersecting in y over R if $\text{ht}(Q_1^* \cap R_1) \leq \text{ht } Q_1^*$ for every prime ideal $Q_1^* \in \text{Spec}(R^*[1/y])$, or, equivalently, if, for every $Q^* \in \text{Spec } R^*$ with $y \notin Q^*$, we have $\text{ht}(Q^* \cap R_1) \leq \text{ht } Q^*$. Thus, to complete the proof of Theorem 3.5, it suffices to prove Claim 3.6.

Claim 3.6. *For every prime ideal $Q^* \in \text{Spec } R^*$ with $y \notin Q^*$, we have*

$$\text{ht}(Q^* \cap R_1) \leq \text{ht } Q^*.$$

Proof of Claim 3.6. Since $\dim R^* = d$ and $y \notin Q^*$, we have $\text{ht } Q^* = r \leq d - 1$. Since the map $R \hookrightarrow R^*$ is flat, we have $\text{ht}(Q^* \cap R) \leq \text{ht } Q^* = r$. Suppose that $Q := Q^* \cap R_1$ has height at least $r + 1$ in $\text{Spec } R_1$. Since R_1 is a localized polynomial ring in one variable over R and $\text{ht}(Q \cap R) \leq r$, we have $\text{ht}(Q) = r + 1$. Let $P := \lambda_0^{-1}(Q) \in \text{Spec } S$. Then $\text{ht } P = r + 1$ and $y \notin P$.

Let $P^* := \lambda_a^{-1}(Q^*)$. Since the prime ideals of S^* that contain $t - a$ and have height $r + 1$ are in one-to-one correspondence with the prime ideals of R^* of height r , we have $\text{ht } P^* = r + 1$. By the commutativity of the diagram, we also have $y \notin P^*$ and $P \subseteq P^* \cap S$, and so

$$r + 1 = \text{ht } P \leq \text{ht}(P^* \cap S) \leq \text{ht } P^* = r + 1,$$

where the last inequality holds because the map $S \hookrightarrow S^*$ is flat. It follows that $P = P^* \cap S$, and so $P^* \in \mathcal{U}$. This contradicts the fact that $t - a \notin P_1^*$ for each $P_1^* \in \mathcal{U}$. Thus we have $\text{ht}(Q^* \cap R_1) \leq r = \text{ht } Q^*$, as asserted in Claim 3.6.

This completes the proof of Theorem 3.5. \square

Theorem 3.5 yields a necessary and sufficient condition for an element of R^* that is algebraically independent over R to be primarily limit-intersecting in y over R .

Remarks 3.7. Assume notation as in Setting 3.4.

- (1) For each $a \in y^2 S^*$ as in Setting 3.4, we have $(t - a)S^* = (t - \tau_a)S^*$. Hence $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\} \iff t - \tau_a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.
 - (2) If $a \in R^*$, then the commutativity of Diagram 3.5.0 implies that $\tau_a = a$.
 - (3) For $\tau \in R^*$, we have $\tau = b_0 + b_1 y + \tau'$, where b_0 and b_1 are in R and $\tau' \in y^2 R^*$.
 - (a) The rings $R[\tau]$ and $R[\tau']$ are equal. Hence τ is primarily limit-intersecting in y over R if and only if τ' is primarily limit-intersecting in y over R .
 - (b) Assume $\tau \in R^*$ is algebraically independent over R . Then τ is primarily limit-intersecting in y over R if and only if $t - \tau' \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.
- Item 3b follows from Theorem 3.5 by setting $a = \tau'$ and applying item 3a and item 2.

We use Theorem 3.5 and Lemma 3.3 to prove Theorem 3.8.

Theorem 3.8. *Let (R, \mathbf{m}) be a countable excellent normal local domain of dimension $d \geq 2$, let y be a nonzero element in \mathbf{m} , and let R^* denote the (y) -adic completion of R . Then there exists an element $\tau \in yR^*$ that is primarily limit-intersecting in y over R .*

Proof. As in Setting 3.4, let

$$\mathcal{U} := \{P^* \in \text{Spec } S^* \mid \text{ht}(P^* \cap S) = \text{ht } P^*, \text{ and } y \notin P^*\}.$$

Since the ring S is countable and Noetherian, the set \mathcal{U} is countable. Lemma 3.3 implies that there exists an element $a \in y^2 S^*$ such that $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$. By Theorem 3.5, the element τ_a is primarily limit-intersecting in y over R . \square

The existence of more primarily limit-intersecting elements.

To establish the existence of more than one primarily limit-intersecting element we use the following setting.

Setting 3.9. Let (R, \mathbf{m}) be a d -dimensional excellent normal local domain, let y be a nonzero element of \mathbf{m} and let R^* denote the (y) -adic completion of R . Let t_1, \dots, t_{n+1} be indeterminates over R , and let S_n and S_{n+1} denote the localized polynomial rings

$$S_n := R[t_1, \dots, t_n]_{(\mathbf{m}, t_1, \dots, t_n)} \quad \text{and} \quad S_{n+1} := R[t_1, \dots, t_{n+1}]_{(\mathbf{m}, t_1, \dots, t_{n+1})}.$$

Let S_n^* denote the I_n -adic completion of S_n , where $I_n := (y, t_1, \dots, t_n)S_n$. Then $S_n^* = R^*[[t_1, \dots, t_n]]$ is a $(d+n)$ -dimensional normal Noetherian local domain with maximal ideal $\mathbf{n}^* = (\mathbf{m}, t_1, \dots, t_n)S_n^*$. Assume that $\tau_1, \dots, \tau_n \in yR^*$ are primarily limit-intersecting in y over R , and define $\lambda : S_n^* \rightarrow R^*$ to be the R^* -algebra homomorphism such that $\lambda(t_i) = \tau_i$, for $1 \leq i \leq n$.

Since $S_n^* = R^*[[t_1 - \tau_1, \dots, t_n - \tau_n]]$, we have $\mathbf{p}_n := \ker \lambda = (t_1 - \tau_1, \dots, t_n - \tau_n)S_n^*$. Consider the commutative diagram:

$$\begin{array}{ccccc} S_n = R[t_1, \dots, t_n]_{(\mathbf{m}, t_1, \dots, t_n)} & \xrightarrow{\subseteq} & S_n^* = R^*[[t_1, \dots, t_n]] & \xrightarrow{\subseteq} & S_n^*[1/y] \\ \lambda_0, \cong \downarrow & & \lambda \downarrow & & \\ R \xrightarrow{\subseteq} R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} & \xrightarrow{\varphi_0} & R^* & \xrightarrow{\alpha} & R^*[1/y]. \end{array}$$

Let S_{n+1}^* denote the I_{n+1} -adic completion of S_{n+1} , where $I_{n+1} := (y, t_1, \dots, t_{n+1})S_{n+1}$. For each element $a \in y^2 S_{n+1}^*$, we have

$$(3.9.1) \quad S_{n+1}^* = S_n^*[[t_{n+1}]] = S_n^*[[t_{n+1} - a]].$$

Let $\lambda_a : S_{n+1}^* \rightarrow R^*$ denote the composition

$$S_{n+1}^* = S_n^*[[t_{n+1}]] \longrightarrow \frac{S_n^*[[t_{n+1}]]}{(t_{n+1} - a)} = S_n^* \xrightarrow{\lambda} R^*,$$

and let $\tau_a := \lambda_a(t_{n+1}) = \lambda_a(a)$. We have $\ker \lambda_a = (\mathbf{p}_n, t_{n+1} - a)S_{n+1}^*$. Consider the commutative diagram

$$\begin{array}{ccccccc} S_n & \xrightarrow{\subseteq} & S_n^* & \xrightarrow{\subseteq} & S_{n+1}^* & \longrightarrow & S_{n+1}^*[1/y] \\ \lambda_0, \cong \downarrow & & \lambda \downarrow & & \lambda_a \downarrow & & \downarrow \\ R \xrightarrow{\subseteq} R_n & \xrightarrow{\varphi_0} & R^* & \xrightarrow{=} & R^* & \longrightarrow & R^*[1/y]. \end{array}$$

Diagram 3.9.2

Let

$$\mathcal{U} := \{P^* \in \text{Spec } S_{n+1}^* \mid P^* \cap S_{n+1} = P, y \notin P \text{ and } P^* \text{ is minimal over } (P, \mathbf{p}_n)S_{n+1}^*\}.$$

Notice that $y \notin P^*$ for each $P^* \in \mathcal{U}$, since $y \in R$ implies $\lambda_a(y) = y$.

Theorem 3.10. *With the notation of Setting 3.9, the elements $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R if and only if $t_{n+1} - a \notin \bigcup \{P^* \mid P^* \in \mathcal{U}\}$.*

Proof. Assume that $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R . Then $\tau_1, \dots, \tau_n, \tau_a$ are algebraically independent over R . Consider the following commutative diagram:

$$\begin{array}{ccc} S_{n+1} = R[t_1, \dots, t_{n+1}]_{(\mathbf{m}, t_1, \dots, t_{n+1})} & \xrightarrow{\subseteq} & S_{n+1}^* = R^*[[t_1, \dots, t_{n+1}]] \\ \lambda_1 \downarrow & & \lambda_a \downarrow \\ R \xrightarrow{\subseteq} R_{n+1} = R[\tau_1, \dots, \tau_a]_{(\mathbf{m}, \tau_1, \dots, \tau_a)} & \longrightarrow & R^*. \end{array}$$

Diagram 3.10.0

The map λ_1 is the restriction of λ_a to S_{n+1} , and is an isomorphism since $\tau_1, \dots, \tau_n, \tau_a$ are algebraically independent over R .

If $t_{n+1} - a \in P^*$ for some $P^* \in \mathcal{U}$, we prove that $\varphi : R_{n+1} \rightarrow R^*[1/y]$ is not flat, a contradiction to our assumption that $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting. Since $P^* \in \mathcal{U}$, we have $\mathbf{p}_n \subset P^*$. Then $t_{n+1} - a \in P^*$ implies $\ker \lambda_a \subset P^*$. Let $\lambda_a(P^*) := Q^*$. Then $\lambda_a^{-1}(Q^*) = P^*$ and $\text{ht } P^* = n + 1 + \text{ht } Q^*$. Since $P^* \in \mathcal{U}$, we have $y \notin P^*$. The commutativity of Diagram 3.10.0 implies that $y \notin Q^*$. Let $P := P^* \cap S_{n+1}$ and let $Q := Q^* \cap R_{n+1}$. Commutativity of Diagram 3.10.0 and λ_0 an isomorphism imply that $\text{ht } P = \text{ht } Q$. Since P^* is a minimal prime of $(P, \mathbf{p}_n)S_{n+1}^*$, \mathbf{p}_n is n -generated, and S_{n+1}^* is Noetherian and catenary, we have $\text{ht } P^* \leq \text{ht } P + n$. Hence $\text{ht } P \geq \text{ht } P^* - n$. Thus

$$\text{ht } Q = \text{ht } P \geq \text{ht } P^* - n = \text{ht } Q^* + n + 1 - n = \text{ht } Q^* + 1.$$

The fact that $\text{ht } Q > \text{ht } Q^*$ implies that the map $R_{n+1} \rightarrow R^*[1/y]$ is not flat. This proves the forward direction.

For the converse, we have

Assumption 3.10.1: $t_{n+1} - a \notin \bigcup \{ P^* \mid P^* \in \mathcal{U} \}.$

Since $\lambda_a : S_{n+1}^* \rightarrow R^*$ is an extension of $\lambda : S_n^* \rightarrow R^*$ as in Diagram 3.9.2, we have $\ker \lambda_a \cap S_n = (0)$. Let $\mathbf{p} := (t_{n+1} - a)S_{n+1}^* = (t_{n+1} - \tau_a)S_{n+1}^*$. As in Equation 3.9.1, we have

$$S_{n+1}^* = R^*[[t_1, \dots, t_{n+1}]] = R^*[[t_1 - \tau_1, \dots, t_n - \tau_n, t_{n+1} - a]].$$

Thus $P^* := (\mathbf{p}_n, \mathbf{p})S_{n+1}^*$ is a prime ideal of height $n+1$ and $P^* \cap R^* = (0)$. It follows that $y \notin P^*$. We show that $P^* \cap S_{n+1} = (0)$. Assume that $P = P^* \cap S_{n+1} \neq (0)$. Since $\text{ht } P^* = n+1$, P^* is minimal over $(P, \mathbf{p}_n)S_{n+1}^*$, and so $P^* \in \mathcal{U}$, a contradiction to Assumption 3.10.1. Therefore $P^* \cap S_{n+1} = (0)$. It follows that $\mathbf{p} \cap S_{n+1} = (0)$ since $\mathbf{p} \subset P^*$. Thus $\ker \lambda_1 = (0)$, and so λ_1 in Diagram 3.10.0 is an isomorphism. Therefore τ_a is algebraically independent over R_n .

Since R is excellent and R_{n+1} is a localized polynomial ring in $n+1$ variables over R , the hypotheses of Corollary 3.2 are satisfied for the composition

$$R \hookrightarrow R_{n+1} \hookrightarrow R^*[1/y].$$

It follows that the elements $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R if, for every $Q^* \in \text{Spec } R^*$ with $y \notin Q^*$, we have $\text{ht}(Q^* \cap R_{n+1}) \leq \text{ht } Q^*$. Thus, to complete the proof of Theorem 3.10 with $\tau_{n+1} = \tau_a$, it suffices to prove Claim 3.11.

Claim 3.11. *Let $Q^* \in \text{Spec } R^*$ with $y \notin Q^*$ and $\text{ht } Q^* = r$. Then*

$$\text{ht}(Q^* \cap R_{n+1}) \leq r.$$

Proof of Claim 3.11. Let $Q_1 := Q^* \cap R_{n+1}$ and let $Q_0 := Q^* \cap R_n$. Suppose $\text{ht } Q_1 > r$. Notice that $r < d$, since $d = \dim R^*$ and $y \notin Q^*$.

Since τ_1, \dots, τ_n are primarily limit-intersecting in y over R , the extension

$$R_n := R[\tau_1, \dots, \tau_n]_{(\mathfrak{m}, \tau_1, \dots, \tau_n)} \hookrightarrow R^*[1/y]$$

from Diagram 3.9.2 is flat. Thus $\text{ht } Q_0 \leq r$ and $\text{ht } Q_0 \leq \text{ht } L^*$ for every prime ideal L^* of R^* with $Q_0 R^* \subseteq L^* \subseteq Q^*$. Since R_{n+1} is a localized polynomial ring in the indeterminate τ_a over R_n , we have that $\text{ht } Q_1 \leq \text{ht } Q_0 + 1 = r + 1$. Thus $\text{ht } Q_1 = r + 1$ and $\text{ht } Q_0 = r$. It follows that Q^* is a minimal prime of $Q_0 R^*$.

Let $h(\tau_a) \in R_n[\tau_a] = R_{n+1}$ be a polynomial in the variable τ_a over the ring R_n such that

$$h(\tau_a) \in (Q^* \cap R_n[\tau_a]) \setminus (Q^* \cap R_n)R_{n+1}.$$

It follows that Q_1 is a minimal prime of the ideal $(Q_0, h(\tau_a))R_{n+1}$.

With notation from Diagram 3.9.2, define

$$P_0 := \lambda_0^{-1}(Q_0) \text{ and } P_0^* := \lambda^{-1}(Q^*).$$

Since λ_0 is an isomorphism, P_0 is a prime ideal of S_n with $\text{ht } P_0 = r$. Moreover, we have the following:

- (1) $P_0^* \cap S_n = P_0$ (by commutativity in Diagram 3.9.2),
- (2) $y \notin P_0^*$ (by item 1),
- (3) P_0^* is a minimal prime of $(P_0, \mathbf{p}_n)S_n^*$ (since $S_n^*/\mathbf{p}_n = R^*$ in Diagram 3.9.2, and Q^* is a minimal prime of $Q_0 R^*$),
- (4) $\text{ht } P_0^* = n + r$ (by the correspondence between prime ideals of S_n^* containing \mathbf{p}_n and prime ideals of R^*).

Consider the commutative diagram below with the left and right ends identified:

$$\begin{array}{ccccccc} S_{n+1}^* & \longleftarrow & S_n^* & \longleftarrow & S_n & \longrightarrow & S_{n+1} \xrightarrow{\theta} S_{n+1}^* \\ \lambda_a \downarrow & & \lambda \downarrow & & \lambda_0, \cong \downarrow & & \lambda_1, \cong \downarrow & & \lambda_a \downarrow \\ R^* & \longleftarrow & R^* & \longleftarrow & R_n & \longrightarrow & R_{n+1} & \longrightarrow & R^*, \end{array}$$

Diagram 3.11.0

where λ, λ_0 and λ_1 are as in Diagrams 3.9.2 and 3.10.0, and so λ_a restricted to S_n^* is λ . Let $h(t_{n+1}) = \lambda_1^{-1}(h(\tau_a))$ and set

$$P_1 := \lambda_1^{-1}(Q_1) \in \text{Spec}(S_{n+1}), \text{ and } P^* := \lambda_a^{-1}(Q^*) \in \text{Spec}(S_{n+1}^*).$$

Then P_1 is a minimal prime of $(P_0, h(t_{n+1}))S_{n+1}$, since Q_1 is a minimal prime of $(Q_0, h(\tau_a))R_{n+1}$. Since $Q_1 \subseteq Q^*$, we have $h(t_{n+1}) \in P^*$ and $P_1 S_{n+1}^* \subseteq P^*$ because $\lambda_a(h(t_{n+1})) = \lambda_1(h(t_{n+1})) = h(\tau_a) \in Q_1$ and $\lambda_a(P_1) = \lambda_1(P_1) = Q_1$. By the correspondence between prime ideals of S_{n+1}^* containing $\ker(\lambda_a) = \mathbf{p}_{n+1}$ and

prime ideals of R^* , we see

$$\text{ht } P^* = \text{ht } Q^* + n + 1 = r + n + 1.$$

Since $\lambda_a(P_0^*) \subseteq Q^*$, we have $P_0^* \subseteq P^*$, but $h(t_{n+1}) \notin P_0$ implies $h(t_{n+1}) \notin P_0^* S_{n+1}^*$. Therefore

$$(P_0, \mathbf{p}_n)S_{n+1}^* \subseteq P_0^* S_{n+1}^* \subsetneq (P_0^*, h(t_{n+1}))S_{n+1}^* \subseteq P^*.$$

By items 3 and 4 above, $\text{ht } P_0^* = n + r$ and P_0^* is a minimal prime of $(P_0, \mathbf{p}_n)S_n^*$. Since $\text{ht } P^* = n + r + 1$, it follows that P^* is a minimal prime of $(P_0, h(t_{n+1}), \mathbf{p}_n)S_{n+1}^*$. Since $(P_0, h(t_{n+1}), \mathbf{p}_n)S_{n+1}^* \subseteq (P_1, \mathbf{p}_n)S_{n+1}^* \subseteq P^*$, we have P^* is a minimal prime of $(P_1, \mathbf{p}_n)S_{n+1}^*$. But then, by Assumption 3.10.1 on \mathcal{U} , we have $t_{n+1} - a \notin P^*$, a contradiction. This contradiction implies that $\text{ht } Q_1 = r$.

This completes the proof of Claim 3.11 and thus the proof of Theorem 3.10. \square

We use Theorem 3.8, Theorem 3.10 and Lemma 3.3 to prove in Theorem 3.12 the existence over a countable excellent normal local domain of dimension at least two of an infinite sequence of primarily limit-intersecting elements.

Theorem 3.12. *Let R be a countable excellent normal local domain of dimension $d \geq 2$, let y be a nonzero element in the maximal ideal \mathbf{m} of R , and let R^* be the (y) -adic completion of R . Let n be a positive integer. Then*

- (1) *If the elements $\tau_1, \dots, \tau_n \in yR^*$ are primarily limit-intersecting in y over R , then there exists an element $\tau_a \in yR^*$ such that $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R .*
- (2) *There exists an infinite sequence $\tau_1, \dots, \tau_n, \dots \in yR^*$ of elements that are primarily limit-intersecting in y over R .*

Proof. By Definition 1.5, item 1 implies item 2; thus it suffices to prove item 1. Theorem 3.8 implies the existence of an element $\tau_1 \in yR^*$ that is primarily limit-intersecting in y over R . As in Setting 3.9, let

$$\mathcal{U} := \{P^* \in \text{Spec } S_{n+1}^* \mid P^* \cap S_{n+1} = P, y \notin P \text{ and } P^* \text{ is minimal over } (P, \mathbf{p}_n)S_{n+1}^*\}.$$

Since the ring S_{n+1} is countable and Noetherian, the set \mathcal{U} is countable. Lemma 3.3 implies that there exists an element $a \in y^2 S_{n+1}^*$ such that

$$t_{n+1} - a \notin \bigcup \{P^* \mid P^* \in \mathcal{U}\}.$$

By Theorem 3.10, the elements $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R . \square

Normal Noetherian domains that are not excellent.

Using Theorem 3.8, we establish in Theorem 3.13, for every countable excellent normal local domain R of dimension $d \geq 2$, the existence of a primarily limit-intersecting element $\eta \in yR^*$ such that the constructed Noetherian domain

$$B = A = R^* \cap \mathcal{Q}(R[\eta])$$

is not a Nagata domain and hence is not excellent.³

Theorem 3.13. *Let R be a countable excellent normal local domain of dimension $d \geq 2$, let y be a nonzero element in the maximal ideal \mathfrak{m} of R , and let R^* be the (y) -adic completion of R . There exists an element $\eta \in yR^*$ such that*

- (1) *η is primarily limit-intersecting in y over R .*
- (2) *The associated intersection domain $A := R^* \cap \mathcal{Q}(R[\eta])$ is equal to its approximation domain B .*
- (3) *The ring A has a height-one prime ideal \mathfrak{p} such that $R^*/\mathfrak{p}R^*$ is not reduced.*

Thus the integral domain $A = B$ associated to η is a normal Noetherian local domain that is not a Nagata domain and hence is not excellent.

Proof. Since $\dim R \geq 2$, there exists $x \in \mathfrak{m}$ such that $\text{ht}(x, y)R = 2$. By Theorem 3.8, there exists $\tau \in yR^*$ such that τ is primarily limit-intersecting in y over R . Hence the extension $R[\tau] \rightarrow R^*[1/y]$ is flat. Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\eta := (x + \tau)^n$. Since τ is algebraically independent over R , the element η is also algebraically independent over R . Moreover, the polynomial ring $R[\tau]$ is a free $R[\eta]$ -module with $1, \tau, \dots, \tau^{n-1}$ as a free module basis. Hence the map $R[\eta] \rightarrow R^*[1/y]$ is flat. It follows that η is primarily limit-intersecting in y over R . Therefore the intersection domain $A := R^* \cap \mathcal{Q}(R[\eta])$ is equal to its associated approximation domain B and is a normal Noetherian domain with (y) -adic completion R^* . Since η is a prime element of the polynomial ring $R[\eta]$ and $B[1/y]$ is a localization of $R[\eta]$, it follows that $\mathfrak{p} := \eta B$ is a height-one prime ideal of B . Since $\tau \in R^*$, and $\eta = (x + \tau)^n$, the ring $R^*/\mathfrak{p}R^*$ contains nonzero nilpotent elements. Since $\widehat{R} = \widehat{B}$ is faithfully flat over R^* , it follows that $\widehat{B}/\mathfrak{p}\widehat{B}$ has nonzero nilpotent elements. Since a Nagata local domain is analytically unramified, it follows that the normal Noetherian domain B is not a Nagata ring, [6, page 264] or [8, (32.2)]. \square

4. OTHER RESULTS AND EXAMPLES USING THE CONSTRUCTION

We use the following notation for the beginning of this section, and make several remarks concerning properties of and relationships among the integral domains being considered.

³“Nagata” is defined after Question 1.2; see also [4, Definitions 2.3.1, 3.28], [6, pages 264, 260].

Notation and Remarks 4.1. Let k be a field, let x and y be indeterminates over k , and let

$$\sigma := \sum_{i=1}^{\infty} a_i x^i \in xk[[x]] \quad \text{and} \quad \tau := \sum_{i=1}^{\infty} b_i y^i \in yk[[y]]$$

be formal power series that are algebraically independent over the fields $k(x)$ and $k(y)$, respectively. Let $R := k[x, y]_{(x, y)}$, and let σ_n, τ_n be the n^{th} endpieces of σ, τ respectively. Define

$$\begin{aligned} (4.1.0) \quad C_n &:= k[x, \sigma_n]_{(x, \sigma_n)}, \quad C := k(x, \sigma) \cap k[[x]] = \varinjlim (C_n) = \bigcup_{n=1}^{\infty} C_n ; \\ D_n &:= k[y, \tau_n]_{(y, \tau_n)}, \quad D := k(y, \tau) \cap k[[y]] = \varinjlim (D_n) = \bigcup_{n=1}^{\infty} D_n ; \\ U_n &:= k[x, y, \sigma_n, \tau_n], \quad U := \varinjlim U_n = \bigcup_{n=1}^{\infty} U_n ; \\ B_n &:= k[x, y, \sigma_n, \tau_n]_{(x, y, \sigma_n, \tau_n)} \quad B := \varinjlim (B_n) = \bigcup_{n=1}^{\infty} B_n ; \\ A &:= k(x, y, \sigma, \tau) \cap k[[x, y]]. \end{aligned}$$

Since $k[[x, y]]$ is the (x, y) -adic completion of the Noetherian ring R , the ring $k[[x, y]] = \widehat{R}$ is faithfully flat over R . Hence we have

$$(x, y)^n k[[x, y]] \cap R = (x, y)^n R$$

for each $n \in \mathbb{N}$.

The relationships

$$\sigma_n = -xa_{n+1} + x\sigma_{n+1} \quad \text{and} \quad \tau_n = -yb_{n+1} + y\tau_{n+1}$$

among the endpieces imply for each positive integer n the inclusions

$$C_n \subset C_{n+1}, \quad D_n \subset D_{n+1}, \quad \text{and} \quad B_n \subset B_{n+1}.$$

Moreover, for each of these inclusions we have birational domination of the larger local ring over the smaller, and the local rings C_n, D_n, B_n are all dominated by $k[[x, y]] = \widehat{R}$.

Since $(x, y, \sigma_n, \tau_n)U_n$ is a maximal ideal of U_n that is contained in $(x, y)U$, a proper ideal of U , it follows that $(x, y)U \cap U_n = (x, y, \sigma_n, \tau_n)U_n$. Since B_n is the localization of the polynomial ring U_n at the maximal ideal $(x, y, \sigma_n, \tau_n)U_n$, we have $(x, y)B \cap B_n = (x, y, \sigma_n, \tau_n)B$ for each $n \in \mathbb{N}$.

We have $\sigma_{n+1} \in U_n[\frac{1}{x}] \subseteq U_n[\frac{1}{xy}]$ and $\tau_{n+1} \in U_n[\frac{1}{y}] \subseteq U_n[\frac{1}{xy}]$, for each $n \in \mathbb{N}$. Hence $U_{n+1} \subseteq U_n[\frac{1}{xy}]$, and $U \subseteq U_n[\frac{1}{xy}]$, and $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$.

The rings C and D are rank-one discrete valuation domains that are directed unions of two-dimensional regular local domains. Each of the rings B_n is a four-dimensional regular local domain that is a localized polynomial ring over the field k . Thus B is the directed union of a chain of four-dimensional regular local domains.

Theorem 4.2. *Assume the setting of Notation 4.1. Then the ring A is a two-dimensional regular local domain that birationally dominates the ring B ; A has maximal ideal $(x, y)A$ and completion $\widehat{A} = k[[x, y]]$. Moreover we have:*

- (1) *The rings U and B are UFDs,*
- (2) *B is a local Krull domain with maximal ideal $\mathbf{n} = (x, y)B$,*
- (3) *The dimension of B is either 2 or 3, depending on the choice of σ and τ ,*
- (4) *B is Hausdorff in the topology defined by the powers of \mathbf{n} ,*
- (5) *The \mathbf{n} -adic completion \widehat{B} of B is canonically isomorphic to $k[[x, y]]$, and*
- (6) *The following statements are equivalent:*
 - (a) $B = A$.
 - (b) B is a two-dimensional regular local domain.
 - (c) B is Noetherian.
 - (d) Every finitely generated ideal of B is closed in the \mathbf{n} -adic topology on B .
 - (e) Every principal ideal of B is closed in the \mathbf{n} -adic topology on B .

Proof. The assertions about A follow from a theorem of Valabrega [12]; see [4, Proposition 4.13]. Since U_0 has field of fractions $k(x, y, \sigma, \tau) = \mathcal{Q}(A)$ and $U_0 \subseteq B \subseteq A$, the extension $B \hookrightarrow A$ is birational. Since B is the directed union of the four-dimensional regular local domains B_n and $(x, y)B \cap B_n = (x, y, \sigma_n, \tau_n)B$ for each $n \in \mathbb{N}$, we see that B is local with maximal ideal $\mathbf{n} = (x, y)B$. Since B and A are both dominated by $k[[x, y]]$, it follows that A dominates B .

To prove that U and B are UFDs, we use that U_n is a polynomial ring over a field and $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$. Hence the ring $U[\frac{1}{xy}]$ is a UFD. For each $n \in \mathbb{N}$, the principal ideals xU_n and yU_n are prime ideals in the polynomial ring U_n . Therefore xU and yU are principal prime ideals of U . Moreover, $U_{xU} = B_{xB}$ and $U_{yU} = B_{yB}$ are DVRs since each is the contraction to the field $k(x, y, \sigma, \tau)$ of the (x) -adic or the (y) -adic valuations of $k[[x, y]]$. A theorem of Nagata [9, Theorem 6.3, p. 21] implies that U is a UFD; see also [4, Theorem 2.9 and Fact 2.11]. Since B is a localization of U , the ring B is a UFD. This completes the proof of items 1 and 2.

Since B is dominated by $k[[x, y]]$, the intersection $\cap_{n=1}^{\infty} \mathbf{n}^n = (0)$, and so B is Hausdorff in the topology defined by the powers of \mathbf{n} . We have local injective maps $R \hookrightarrow B \hookrightarrow \widehat{R}$, and for each positive integer n , we have $\mathbf{m}^n B = \mathbf{n}^n$, $\mathbf{m}^n \widehat{R} = \widehat{\mathbf{m}}^n$ and $\widehat{\mathbf{m}}^n \cap R = \mathbf{m}^n$. Since the natural map $R/\mathbf{m}^n \rightarrow \widehat{R}/\mathbf{m}^n \widehat{R} = \widehat{R}/\widehat{\mathbf{m}}^n$ is an isomorphism, the map $R/\mathbf{m}^n \rightarrow B/\mathbf{m}^n B = B/\mathbf{n}^n$ is injective and the map $B/\mathbf{n}^n \rightarrow$

$\widehat{R}/\mathbf{n}^n\widehat{R} = \widehat{R}/\widehat{\mathbf{m}}^n$ is surjective. Since B/\mathbf{n}^n has finite length as an R -module, it follows that $R/\mathbf{m}^n \cong B/\mathbf{n}^n \cong \widehat{R}/\widehat{\mathbf{m}}^n$ for each $n \in \mathbb{N}$ and hence $\widehat{B} = \widehat{R} = k[[x, y]]$. Notice that B is a birational extension of the three-dimensional Noetherian domain $C[y, \tau]$. The dimension of B is at most 3 by a theorem of Cohen, [6, Theorem 15.5] or [4, Theorem 2.9]. This completes the proof of items 3, 4 and 5.

For item 6, since A is a two-dimensional regular local ring, $(a) \implies (b)$. Clearly $(b) \implies (c)$. Since B is local by item 2, and since the completion of a Noetherian local ring is a faithfully flat extension, we have $(c) \implies (d)$. It is clear that $(d) \implies (e)$. To complete the proof of Theorem 4.2, it suffices to show that $(e) \implies (a)$. Since A birationally dominates B , we have $B = A$ if and only if $bA \cap B = bB$ for every element $b \in \mathbf{n}$. The principal ideal bB is closed in the \mathbf{n} -adic topology on B if and only if $bB = b\widehat{B} \cap B$. Also $\widehat{B} = \widehat{A}$ and $bA = b\widehat{A} \cap A$, for every $b \in B$. Thus (e) implies, for every $b \in B$,

$$bB = b\widehat{B} \cap B = b\widehat{A} \cap B = b\widehat{A} \cap A \cap B = bA \cap B,$$

and so $B = A$. This completes the proof of Theorem 4.2. \square

Depending on the choice of σ and τ , the ring B may fail to be Noetherian. Example 4.3 shows that in the setting of Theorem 4.2 the ring B can be strictly smaller than $A := k(x, y, \sigma, \tau) \cap k[[x, y]]$.

Example 4.3. Using the setting of Notation 4.1, let $\tau \in k[[y]]$ be defined to be $\sigma(y)$, that is, set $b_i := a_i$ for every $i \in \mathbb{N}$. We then have that $\theta := \frac{\sigma - \tau}{x - y} \in A$. Indeed,

$$\sigma - \tau = a_1(x - y) + a_2(x^2 - y^2) + \cdots + a_n(x^n - y^n) + \cdots,$$

and so $\theta = \frac{\sigma - \tau}{x - y} \in k[[x, y]] \cap k(x, y, \sigma, \tau) = A$. As a specific example, one may take $k := \mathbb{Q}$ and set $\sigma := e^x - 1$ and $\tau := e^y - 1$. The ring B is a localization of the ring $U := \bigcup_{n \in \mathbb{N}} k[x, y, \sigma_n, \tau_n]$.

Claim 4.4. *The element θ is not in B .*

Proof. If θ is an element of B , then

$$\sigma - \tau \in (x - y)B \cap U = (x - y)U.$$

Let $S := k[x, y, \sigma, \tau]$ and let $U_n := k[x, y, \sigma_n, \tau_n]$ for each positive integer n . We have

$$U = \bigcup_{n \in \mathbb{N}} U_n \subseteq S\left[\frac{1}{xy}\right] \subset S_{(x-y)S},$$

where the last inclusion is because $xy \notin (x - y)S$. Thus $\theta \in B$ implies that

$$\sigma - \tau \in (x - y)S_{(x-y)S} \cap S = (x - y)S,$$

but this contradicts the fact that x, y, σ, τ are algebraically independent over k , and thus S is a polynomial ring over k in x, y, σ, τ . \square

Therefore $\frac{\sigma-\tau}{x-y} \notin B$, and so $B \subsetneq A$ and $(x-y)B \subsetneq (x-y)A \cap B$. Since an ideal of B is closed in the \mathbf{n} -adic topology if and only if the ideal is contracted from \widehat{B} and since $\widehat{B} = \widehat{A}$, the principal ideal $(x-y)B$ is not closed in the \mathbf{n} -adic topology on B . Using Theorem 4.2, we conclude that B is a non-Noetherian three-dimensional local Krull domain having a two-generated maximal ideal such that B birationally dominates a four-dimensional regular local domain. In this connection, also see [4, Example 8.11].

The setting of (4.1) is balanced in x and y in the sense that the roles of x and y are interchangeable. A more truly iterative process is described in Setting 4.5.

Setting 4.5. Let k be a field, let x be an indeterminate over k , and let

$$\sigma := \sum_{i=1}^{\infty} a_i x^i \in k[[x]] \quad \text{with each} \quad a_i \in k$$

be a formal power series that is algebraically independent over the field $k(x)$. As in Example 2.1, let σ_n be the n^{th} endpiece of σ and define

$$C_n := k[x, \sigma_n]_{(x, \sigma_n)} \quad \text{and} \quad C := k(x, \sigma) \cap k[[x]] = \varinjlim_{\rightarrow} (C_n) = \bigcup_{n=1}^{\infty} C_n.$$

Let y be an indeterminate over C and let

$$\tau := \sum_{i=1}^{\infty} b_i y^i \in C[[y]] \quad \text{with each} \quad b_i \in C$$

be a formal power series that is algebraically independent over $C[y]$. Notice that as a special case we may have each $b_i \in k$. Let τ_n be the n^{th} endpiece of τ and define

$$\begin{aligned} (4.5.0) \quad U_n &:= k[x, y, \sigma_n, \tau_n], \quad U := \varinjlim_{\rightarrow} U_n = \bigcup_{n=1}^{\infty} U_n; \\ B_n &:= k[x, y, \sigma_n, \tau_n]_{(x, y, \sigma_n, \tau_n)} \quad B := \varinjlim_{\rightarrow} (B_n) = \bigcup_{n=1}^{\infty} B_n; \\ D_n &:= C[y, \tau_n]_{(x, y, \tau_n)} \quad D := \varinjlim_{\rightarrow} (D_n) = \bigcup_{n=1}^{\infty} D_n; \\ A &:= k(x, y, \sigma, \tau) \cap k[[x, y]]. \end{aligned}$$

Notice that each $U_n \subset k[[x, y]]$ and U_n is a polynomial ring in x, y, σ_n, τ_n over the field k . Since $C \subset B$, we have $B = D$.

Remark 4.6. Let the notation be as in Setting 4.5. For certain choices of σ and τ , the ring B is Noetherian with $B = A$. Let k be the field \mathbb{Q} of rational numbers. Thus $R := \mathbb{Q}[x, y]_{(x, y)}$ is the localized polynomial ring in the variables x and y , and the completion \widehat{R} of R with respect to its maximal ideal $\mathbf{m} := (x, y)R$ is $\widehat{R} = \mathbb{Q}[[x, y]]$, the formal power series ring in x and y . Let $\sigma := e^x - 1 \in \mathbb{Q}[[x]]$,

and $C := \mathbb{Q}[[x]] \cap \mathbb{Q}(x, \sigma)$. Thus C is an excellent DVR with maximal ideal $x C$, and $T := C[y]_{(x, y)C[y]}$ is an excellent countable two-dimensional regular local ring with maximal ideal $(x, y)T$ and with (y) -adic completion $C[[y]]$. The UFD $C[[y]]$ has maximal ideal $\mathfrak{n} = (x, y)$. Since T is countable, Theorem 3.8 implies that there exists $\tau \in C[[y]]$ that is primarily limit-intersecting in y over R . Hence for this choice of $\sigma \in \mathbb{Q}[[x]]$ and $\tau \in C[[y]]$, we have $A = \mathbb{Q}(x, y, \sigma, \tau) \cap C[[y]]$ is Noetherian and equal to its approximation domain $D = B$.

To fit the setting of Notation 4.1 with $k = \mathbb{Q}$, one wants $\tau \in \mathbb{Q}[[y]]$ rather than $\tau \in C[[y]]$. An example with this more restrictive property is given in [4, Example 7.15].

Weakly flat extensions that are not flat.

Let d be an integer with $d \geq 2$. We obtain in Theorem 4.7 extensions that satisfy LF_{d-1} but do not satisfy LF_d ; see Definition 2.7.3. Thus we obtain examples where the intersection domain A is equal to its approximation domain B , but A is not Noetherian.

Theorem 4.7. *Let (R, \mathfrak{m}) be a countable excellent normal local domain. Assume that $\dim R = d + 1 \geq 3$, that $(x_1, \dots, x_d, y)R$ is an \mathfrak{m} -primary ideal, and that R^* is the (y) -adic completion of R . Then there exists $f \in yR^*$ such that f is algebraically independent over R and the map $\varphi : R[f] \rightarrow R^*[1/y]$ is weakly flat but not flat. Indeed, φ satisfies LF_{d-1} , but fails to satisfy LF_d . Thus the intersection domain $A := \mathcal{Q}(R[f]) \cap R^*$ is equal to its approximation domain B , but A is not Noetherian.*

Proof. By Theorem 3.12, there exist elements $\tau_1, \dots, \tau_d \in yR^*$ that are primarily limit-intersecting in y over R . Let

$$f := x_1\tau_1 + \dots + x_d\tau_d.$$

Using that τ_1, \dots, τ_d are algebraically independent over R , we regard f as a polynomial in the polynomial ring $T := R[\tau_1, \dots, \tau_d]$. Let $S := R[f]$. For $Q \in \text{Spec } R^*[1/y]$ and $P := Q \cap T$, consider the composition φ_Q

$$S \longrightarrow T_P \longrightarrow R^*[1/y]_Q.$$

Since τ_1, \dots, τ_d are primarily limit-intersecting in y over R , the map $T \hookrightarrow R^*[1/y]$ is flat. Thus the map φ_Q is flat if and only if the map $S \rightarrow T_P$ is flat. Let $\mathfrak{p} := P \cap R$.

Assume that P is a minimal prime of $(x_1, \dots, x_d)T$. Then \mathfrak{p} is a minimal prime of $(x_1, \dots, x_d)R$. Since T is a polynomial ring over R , we have $P = \mathfrak{p}T$ and $\text{ht}(\mathfrak{p}) = d = \text{ht } P$. Notice that $(\mathfrak{p}, f)S = P \cap S$ and $\text{ht}(\mathfrak{p}, f)S = d + 1$. Since a flat extension satisfies the going-down property, the map $S \rightarrow T_P$ is not flat. Hence φ does not satisfy LF_d .

Assume that $\text{ht } P \leq d - 1$. Then $(x_1, \dots, x_d)T$ is not contained in P . Hence $(x_1, \dots, x_d)R$ is not contained in \mathfrak{p} . Consider the sequence

$$S = R[f] \hookrightarrow R_{\mathfrak{p}}[f] \xrightarrow{\psi} R_{\mathfrak{p}}[\tau_1, \dots, \tau_d] \hookrightarrow T_P,$$

where the first and last injections are localizations. Since the nonconstant coefficients of f generate the unit ideal of $R_{\mathfrak{p}}$, the map ψ is flat; see [4, Theorem 11.20]. Thus φ satisfies LF_{d-1} .

We conclude that the intersection domain $A = R^* \cap \mathcal{Q}(R[f])$ is equal to its approximation domain B and is not Noetherian. \square

We describe a specific example of Theorem 4.7:

Example 4.8. Let d be an integer with $d \geq 2$ and let x_1, \dots, x_d, y be indeterminates over a countable field k . Let R be the localized polynomial ring in the variables x_1, \dots, x_d, y and let R^* be the (y) -adic completion of R . Thus

$$R = k[x_1, \dots, x_d, y]_{(x_1, \dots, x_d, y)} \quad \text{and} \quad R^* = k[x_1, \dots, x_d]_{(x_1, \dots, x_d)}[[y]].$$

As in Theorem 4.7, there exist elements $\tau_1, \dots, \tau_d \in yR^*$ that are primarily limit-intersecting in y over R , and we consider $f := x_1\tau_1 + \dots + x_d\tau_d$. By Theorem 4.7, the map $S \rightarrow R^*[1/y]$ satisfies LF_{d-1} , but does not satisfy LF_d . Thus the intersection domain $A = R^* \cap \mathcal{Q}(R[f])$ is equal to its approximation domain B and is not Noetherian.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

E-mail address: `heinzer@math.purdue.edu`

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824-1027

E-mail address: `rotthaus@math.msu.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NE 68588-0130

E-mail address: `swiegand@math.unl.edu`